Abelian Gauge Invariance

QED is constructed to be invariant under the gauge symmetry

$$\psi(x) \longrightarrow e^{i\alpha(x)}\,\psi(x)$$

Phase depends on x !

In this situation, the ordinary derivative no longer makes sense

$$n^\mu \partial_\mu \psi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[\psi(x+n\varepsilon) - \psi(x)\right]$$

different symm. transformation !

To define a meaningful derivative we need a connection, which compensates the phase difference. For a finite difference, this can be done with a link field

$$U(y,x) \rightarrow e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)} . \quad (\square)$$

with this, one defines the covariant derivative

$$n^\mu D_\mu \psi(x) = \frac{1}{\varepsilon}\left[\psi(x+n\varepsilon) - U(x+n\varepsilon, x)\psi(x)\right] \quad (*)$$

[For lattice gauge theories, one works with $\varepsilon = a =$ "lattice spacing". In this case the entire theory is formulated with a link field and the fermion.]

This transforms like $\psi(x)$. For $\varepsilon \to 0$, we can expand

connection

$$\mathcal{U}(x+\varepsilon n, x) = 1 - ie\,\varepsilon\, n^\mu A_\mu + O(\varepsilon^2)$$

inserting into (*) gives

$$D_\mu \psi = \partial_\mu \psi + ie A_\mu .$$

Inserting into ($\square$) and expanding gives

$$\mathcal{U}(x+\varepsilon n, x) = 1 - ie\varepsilon n^\mu A_\mu + O(\varepsilon^2)$$

$$\longrightarrow (1 + i\varepsilon n^\mu \partial_\mu \alpha)(1 - ie\varepsilon n^\mu A_\mu)$$

$$\left[\ \alpha(x+\varepsilon n) = \alpha(x) + \varepsilon n^\mu \partial_\mu \alpha \right.$$

so $A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu \alpha$

With these ingredients, one obtains a gauge invariant Fermion Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi$$

We can also construct a kinetic term by considering

$$[D_\mu, D_\nu]\Psi(x) \to e^{i\alpha(x)}[D_\mu, D_\nu]\Psi(x)$$

$$[D_\mu, D_\nu]\Psi = ie\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}}\Psi$$

Note

$$F_{\mu\nu} \to e^{-i\alpha(x)}[D_\mu, D_\nu]e^{+i\alpha(x)} = F_{\mu\nu}$$

is gauge invariant.

With this, we can construct a kinetic term:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Chosen so that the kinetic term has canonical normalization.

Lie groups & Lie algebras

The gauge transformations $g(\alpha) = e^{i\alpha}$ form a Lie Group, a continous group.

It is interesting to consider group elements close to unity:

$$g(\alpha) = 1 + i\alpha + O(\alpha^2)$$

and we can reconstruct finite transformations by repeatedly performing small transformations.

$$g(\alpha) = \lim_{n \to \infty} \left(1 + \frac{i\alpha}{n}\right)^n = e^{i\alpha}$$

(*)

We want to extend the concept of gauge invariance to other compact Lie groups. We consider transformations

$$\psi \longrightarrow g(\alpha)\,\psi$$

where $\psi = (\psi_1, \psi_2, \ldots, \psi_N)^T$ and $g(\alpha)$ is a unitary $N \times N$ matrix. A small transformation then takes the form

$$g(\alpha) = \mathbb{1} + i\alpha^a T^a + O(\alpha^2) \qquad a = 1 \ldots d$$

The matrices T^a are called the generators of the group.

Using (*), we obtain finite transformations as

$$g(\alpha) = e^{i\alpha^a T^a}$$

Group multiplication is

$$g(\alpha)\,g(\beta) = e^{i\alpha^a T^a}\, e^{i\beta^b T^b}$$

More commutators

$$= e^{i(\alpha^a+\beta^a)T^a + \frac{1}{2}[T^a,T^b]\alpha^a\beta^b + \cdots\cdots}$$

Baker-Campbell-Hausdorff

If we know the commutators

$$[T^a, T^b] = i f^{abc} T^c \qquad (**)$$

Structure constants

we know the structure of the group.

The vector space of the generators with (**) is called the Lie Algebra.

Because of the close correspondence of the algebra & the group, physicists often use the same symbol, e.g $SU(N)$, for both. Mathematicians use lower-case names $su(N)$ for the algebra.

The commutator relation

$$[T^a,[T^b,T^c]]+[T^b,[T^c,T^a]]+[T^c,[T^a,T^b]]=0$$

imply the Jacobi-identity

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

Killing and Cartan have classified all possible compact simple Lie algebras.

no normal subgroups

(no commuting subset of generators)

They are:

- $SU(N)$ preserves $\chi_a^+ \psi_a$

  $d = N^2 - 1$

- $SO(N)$ preserves $\chi_a \psi_a$

  $N(N-1)/2$

  N-dim vectors

- $Sp(N)$ preserves $\chi_a E_{ab} \psi_b$

  $N \cdot (N+1)/2$

$$E = \begin{pmatrix} 0 & \mathbb{1}_{N/2} \\ -\mathbb{1}_{N/2} & 0 \end{pmatrix}$$

and five exceptional groups

G_2, F_4, E_6, E_7, E_8

14 52 78 133 248

Representations

From QT, you know that the Lie group $SU(2) \cong O(3)$ can be represented differently (different spins!).

Also for the gauge symmetry group, we can consider different representations.

If we have the fields transform as a finite-dim. unitary transformation, the generators will be represented by $d \times d$ hermitian matrices t^a_R

↖ Representation R.

We will only consider irreducible reps., where the matrices cannot be block diagonalized. One can choose the matrices such that

$$\mathrm{tr}\left[t_R^a \, t_R^b \right] = T_R \, \delta^{ab}$$

where T_R is a constant.

For $SU(N)$, the fundamental representation is N-dimensional and for $SU(2)$, we choose

✓ Pauli matrix

$$t^a \equiv t_F^a = \frac{\sigma^a}{2}$$

$$\Rightarrow \quad T_F = \frac{1}{4} \mathrm{tr}[\mathbb{1}] = \frac{1}{2} .$$

We use the <u>same</u> normalization for all N.

For our discussion in QCD, two more representations will be important.

1.) The Conjugate representation:

if $\psi \longrightarrow (1 + i\alpha^a t^a_R)\psi$

then

$$\psi^* \longrightarrow (1 - i\alpha^a (t^a_R)^*)\psi^*$$

$$\doteq (1 + i\alpha^a t^a_{\bar{R}})\psi^*$$

Note: $\psi^*_k (t^a_R)^\dagger_{ki} = \psi^*_k (t_R)_{ki}$

$= (t^T_R)_{ik}\psi^*_k = (t^*_R)_{ik}\psi^*_k$

2.) The adjoint representation

$$(t^a_A)_{bc} = -i f_{abc}$$

The commutation relation

$$[t_A^a, t_A^b] = i f^{abc} T_A^c$$

is just a rewriting of the Jacobi identity.

As for Dirac matrices, it is usually not necessary (& even a bad idea) to work with explicit representations. Below we will discuss two strategies to evaluate group theory factors which arise in Feynman diagrams.

Non-Abelian Gauge Invariance

with the above group theory knowledge, we now construct Yang-Mills theory, the nonabelian generalization of QED. The transformation of the fermion field $\psi = (\psi_1, \ldots, \psi_N)^T$

$$\psi(x) \rightarrow V(x)\psi(x)$$

Lie group

$$V(x) = \exp\left(i\alpha^a(x)\, t^a\right) \in G$$

To define the covariant derivative, we expand the link field

$$U(y,x) \longrightarrow V(y)\,U(y,x)\,V^{\dagger}(x) \qquad (*)$$

as

$$U(x+\varepsilon n, x) = 1 + ig\,\varepsilon\, n^{\mu} A_{\mu}^{a}(x)\, t^{a} \qquad (**)$$

$$D_{\mu} = \partial_{\mu} - ig A_{\mu}^{a} t^{a}$$

Attention, different sign convention than in QED!

Plugging (**) into (*) and expanding in ε, one finds (exercise)

$$A_\mu(x) \longrightarrow V(x)\left[A_\mu(x) + \frac{i}{g}\partial_\mu \right] V^\dagger(x),$$

where we used the short-hand notation

$$A_\mu(x) \equiv A_\mu^a(x)\, t^a \,,$$

consistent with

$$D_\mu \psi \longrightarrow V D_\mu \psi = V D_\mu V^\dagger V \psi$$

To get the kinetic term, we again consider $[D_\mu , D_\nu]\psi$.

One again finds that this quantity does not involve a derivative on the field, but in contrast to QED, there is a commutator term:

$$F_{\mu\nu} \equiv F^a_{\mu\nu}\, t^a := \frac{i}{g}\left[D_\mu , D_\nu\right]$$

$$= \partial_\mu A_\nu - \partial_\mu A_\nu - ig\left[\underset{A^b_\mu t^b}{\underset{\uparrow}{A_\mu}} , \underset{A^c_\nu t^c}{\underset{\uparrow}{A_\nu}}\right]$$

$$= \left(\partial_\mu A^a_\nu - \partial_\mu A^a_\nu\right) t^a$$

$$+ g f^{abc} A^b_\mu A^c_\nu\, t^a$$

Note: $F_{\mu\nu} \longrightarrow V F_{\mu\nu} V^\dagger$ is <u>not</u> gauge invariant! Kinetic term is

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{tr}\left[F_{\mu\nu} F^{\mu\nu} \right]$$

$$= -\frac{1}{2} \mathrm{tr}\left[F^a_{\mu\nu} t^a F^{\mu\nu, b} t^b \right]$$

$$= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

$\mathrm{tr}[t^a t^b] = \frac{1}{2} \delta^{ab}$

Side remark: As mentioned in the introduction, there is a second term

Θ-term breaks $\cancel{P}, \cancel{T}, \cancel{CP}$

$$\Delta\mathcal{L} = +\theta \frac{g^2}{16\pi^2} \mathrm{tr}\left[G_{\mu\nu} \tilde{G}^{\mu\nu} \right]$$

$\tilde{G}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$

which has quite interesting properties.

- total derivative, i.e. boundary term in the action. No effect on EOM. Since A_μ^a is not a physical field, it does not have to vanish at ∞.
- Term represents topological property of fields. Not visible in perturbation theory.

Since we do not observe strong CP violation (e.g. a neutron EDM) θ must be small. It is not understood why this is the case ("strong CP problem").